# Solving Rational Equations

## Exercises

1. Solve each of the following equations algebraically.
   a. $\dfrac{x^2-5x-6}{2x^2-x-3}=0$
   b. $\dfrac{2}{x+2}=\dfrac{2x+1}{x+1}$
   c. $\dfrac{7-x}{x}=\dfrac{x-3}{x+2}$
   d. $\dfrac{x}{x-2}+\dfrac{1}{1-x}=\dfrac{x}{x^2-3x+2}$
   e. $\dfrac{2}{x-3}+\dfrac{3}{x+2}-\dfrac{5}{x+1}=0$
2. Determine the roots of the following equations.
   a. $\dfrac{x^2+x+4}{2x+1}=\dfrac{4}{x}$
   b. $\dfrac{2x^2-x}{3}-4=\dfrac{3}{x}$
3. For each of the following pairs of rational functions, determine the point(s) of intersection of the two functions and illustrate the situation graphically.
   a. $f(x)=\dfrac{2}{x}-3$ and $g(x)=\dfrac{5}{x}$
   b. $f(x)=\dfrac{3}{x+4}$ and $g(x)=\dfrac{2}{x-1}$
   c. $f(x)=\dfrac{x}{x+7}$ and $g(x)=\dfrac{-x}{x-2}$
   d. $f(x)=\dfrac{8}{x-2}$ and $g(x)=\dfrac{4+3x}{x}$
4. The denominator of a fraction is $2$ more than twice the numerator. If $7$ is added to both the numerator and the denominator the new fraction would be equivalent to $\frac{2}{3}$. What is the original fraction?
5. Working together, Dan and Dory can clean their house in $6$ hours. On her own, Dory can clean the house in $10$ hours. How long does it take Dan to clean the house without any help from Dory?
6. Anton has made $40\%$ of his free-throw shots, after attempting $20$ shots at a free-throw competition. How many more consecutive shots must he make in order to increase his percentage to $50\%$?
7. How many litres of water must be added to $0.5\text{ L}$ of a $30\%$ salt solution to obtain a $20\%$ salt solution?
8. It takes Karin $8$ hours to kayak $18\text{ km}$ up river and $18\text{ km}$ back down the river. If the river is flowing at $3\text{ km/h}$, how fast did she paddle assuming she paddled at a constant speed?
9. An Italian tour company provides a bus tour of the Amalfi Coast for a maximum of $32$ people at a total cost of $\$1200$. A local site-seeing group decide to book the tour. The organizer of the group determines that the cost per person will be reduced by $\$15$ if four more people were to join the tour. How many people initially signed up for the bus tour?
10. a. In mathematics, two positive numbers are in a golden ratio if the ratio of the larger number to the smaller number is equal to the ratio of their sum to the larger number. If $5$ and $n$ are in a golden ratio, where $n>5$, find the exact value of $n$.
    b. Two positive numbers are in the silver ratio if the ratio of the larger number to the smaller number is equal to the ratio of the sum of the smaller number plus twice the larger number to the larger number. If $5$ and $n$ are in a silver ratio, where $n>5$, find the exact value of $n$.
11. Given a rational function $f(x)=\dfrac{x^2-x-12}{2x^2+9x+4}$, determine all asymptotes of the function. Show, algebraically, that the graph of the function will cross the horizontal asymptote.
12. Identify all restrictions on the variable, $x$, and solve each equation.
    a. $\dfrac{x+3}{1-\dfrac{3}{1-\dfrac{1}{x+3}}}=-\dfrac{x}{2}$
    b. $\dfrac{1}{x^2-x}+\dfrac{1}{x^2-3x+2}+\dfrac{1}{x^2-5x+6}+\dfrac{1}{x^2-7x+12}=\dfrac{1}{3}$
13. Maggie is driving to Kamloops, a distance of $450\text{ km}$ from her home. Her car burns fuel at a rate of $\left(0.0007x+\dfrac{4}{x}\right)\text{ L/km}$ when driving at a constant speed of $x$ km/h. If gasoline costs $\$1.20\text{/L}$, at approximately what speed should Maggie travel to keep the cost of the trip to $\$62.00$? (Note: speed limits restrict speeds to a maximum of $100\text{ km/h}$)

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## Partial Solutions

1. There is no solution provided for this question.

2. a. First determine the restrictions on $x$. For the equation $\dfrac{x^2+x+4}{2x+1} = \dfrac{4}{x}$, $2x+1 \neq 0$ and $x \neq 0$ so $x \neq -\frac{1}{2}, 0$. Multiply both sides of the equation by the LCD (lowest common denominator), $x(2x+1)$, and solve the resulting equation.

$$\begin{aligned} x(2x+1)\left(\frac{x^2+x+4}{2x+1}\right) &= x(2x+1)\left(\frac{4}{x}\right) \\ x(x^2+x+4) &= 4(2x+1) \\ x^3+x^2+4x &= 8x+4 \\ x^3+x^2-4x-4 &= 0 \\ x^2(x+1)-4(x+1) &= 0 \\ (x+1)(x^2-4) &= 0 \\ (x+1)(x-2)(x+2) &= 0 \end{aligned}$$

Therefore, $x = -2, -1, 2$, all of which satisfy $x \neq -\frac{1}{2}, 0$.

b. First determine the restrictions on $x$. For the equation $\dfrac{2x^2-x}{3} - 4 = \dfrac{3}{x}$, $x \neq 0$. Multiply both sides by the LCD, $3x$, and solve the resulting equation.

$$\begin{aligned} 3x\left(\frac{2x^2-x}{3} - 4\right) &= 3x\left(\frac{3}{x}\right) \\ x(2x^2-x)-4(3x) &= 3(3) \\ 2x^3-x^2-12x &= 9 \\ 2x^3-x^2-12x-9 &= 0 \end{aligned}$$

By the factor theorem, if $f(x) = 2x^3-x^2-12x-9$ and $f(-1) = 0$, then $(x+1)$ is a factor.

$$\begin{array}{r|rrrr} & 2x^2 & -\ 3x & -\ 9 & \\ \hline x+1 & 2x^3 & -\ x^2 & -\ 12x & -\ 9 \\ & -\ (2x^3 & +\ 2x^2) & & \\ \hline & & -3x^2 & -\ 12x & \\ & & -\ (-3x^2 & -\ 3x) & \\ \hline & & & -9x & -\ 9 \\ & & & -\ (-9x & -\ 9) \\ \hline & & & & 0 \end{array}$$

So, $f(x) = 2x^3-x^2-12x-9 = (x+1)(2x^2-3x-9) = (x+1)(x-3)(2x+3) = 0$.
Therefore, $x = -\frac{3}{2}, -1, 3$, all of which satisfy $x \neq 0$.

3. There is no solution provided for this question.

4. Let $x$ represent the value in the numerator and $y$ represent the value in the denominator of the fraction. Now,

$$y = 2x+2 \tag{1}$$

and

$$\frac{x+7}{y+7} = \frac{2}{3} \tag{2}$$

Substituting $(1)$ into $(2)$,

$$\begin{aligned} \frac{x+7}{(2x+2)+7} &= \frac{2}{3} \\ \frac{x+7}{2x+9} &= \frac{2}{3} \\ 3x+21 &= 4x+18 \\ \therefore x &= 3 \\ \therefore y &= 2(3)+2 = 8 \end{aligned}$$

Therefore, the original fraction is $\frac{3}{8}$.

5. There is no solution provided for this question.

6. Let $x$ represent the number of consecutive free-throw shots Anton must make. Since Anton has made $40\%$ of his free-throw shots, after attempting $20$, he has made $8$ shots. To determine the percent of shots made we can use the formula:

$$\text{Percent} = \frac{\text{Successful Free-Throws}}{\text{Total Free-Throws}} \times 100$$

Then,

$$\begin{aligned} 50 &= \frac{8+x}{20+x} \times 100 \\ 0.5 &= \frac{8+x}{20+x} \\ 10 + 0.5x &= 8 + x \\ \therefore x &= 4 \end{aligned}$$

Therefore, Anton must make $4$ consecutive free-throw shots to increase his percentage to $50\%$.

7. There is no solution provided for this question.

8. Let $x$ represent the speed, in $\text{km/h}$, that Karin paddled. When Karin was paddling upstream, the current was travelling in the opposite direction so the speed at which she travelled upstream was $(x-3)$ km/h. Similarly, when Karin paddled downstream, the current was travelling in the same direction so the speed at which she travelled downstream was $(x+3)$ km/h. Note that $x > 3$ in order for Karin to travel upstream $18\ \text{km}$. Now,

$$\begin{aligned} \text{Total Time} &= \text{Time Upstream} + \text{Time Downstream} \\ \text{Total Time} &= \frac{\text{Distance Upstream}}{\text{Speed Upstream}} + \frac{\text{Distance Downstream}}{\text{Speed Downstream}} \\ 8 &= \frac{18}{x-3} + \frac{18}{x+3}, x > 3 \\ 8(x-3)(x+3) &= \frac{18}{x-3}(x-3)(x+3) + \frac{18}{x+3}(x-3)(x+3) \\ 8(x^2-9) &= 18(x+3) + 18(x-3) \\ 8x^2 - 72 &= 36x \\ 8x^2 - 36x - 72 &= 0 \\ 2x^2 - 9x - 18 &= 0 \\ (2x+3)(x-6) &= 0 \end{aligned}$$

$x = \frac{-3}{2}$ or $x = 6$, but $x > 3$ therefore $x = 6\ \text{km/h}$. Karin paddled at a speed of $6\ \text{km/h}$.

9. There is no solution provided for this question.

10. a. Equating the ratios and representing them as fractions,

$$\begin{aligned} n : 5 &= (5+n) : n \\ \frac{n}{5} &= \frac{5+n}{n} \\ n^2 &= 25 + 5n \\ n^2 - 5n - 25 &= 0 \\ n &= \frac{5 \pm \sqrt{(-5)^2 - 4(1)(-25)}}{2(1)} \\ &= \frac{5 \pm \sqrt{125}}{2} \\ &= \frac{5 \pm 5\sqrt{5}}{2} \\ &= \frac{5(1 \pm \sqrt{5})}{2} \end{aligned}$$

But $n > 0$ so $n = \frac{5(1+\sqrt{5})}{2}$.

b.

$$\begin{aligned} n : 5 &= (5+2n) : n \\ \frac{n}{5} &= \frac{5+2n}{n} \\ n^2 &= 25 + 10n \\ n^2 - 10n - 25 &= 0 \\ n &= \frac{10 \pm \sqrt{(-10)^2 - 4(1)(-25)}}{2(1)} \\ &= \frac{10 \pm \sqrt{200}}{2} \\ &= \frac{10 \pm 10\sqrt{2}}{2} \\ &= 5 \pm 5\sqrt{2} \end{aligned}$$

But $n > 0$, so $n = 5 + 5\sqrt{2}$.

11. There is no solution provided for this question.

12. a. Determining the restrictions on $x$ for the equation $\dfrac{x+3}{1 - \dfrac{3}{1 - \dfrac{1}{x+3}}} = -\dfrac{x}{2}$,

$$\begin{aligned} x+3 &\neq 0 \\ x &\neq -3 \end{aligned}$$

and

$$\begin{aligned} 1-\frac{1}{x+3} &\neq 0 \\ \frac{(x+3)-1}{x+3} &\neq 0 \\ x+2 &\neq 0 \\ x &\neq -2 \end{aligned}$$

and

$$\begin{aligned} 1-\frac{3}{1-\frac{1}{x+3}} &\neq 0 \\ 1-\frac{3(x+3)}{(1-\frac{1}{x+3})(x+3)} &\neq 0 \\ \frac{(x+2)-(3x+9)}{x+2} &\neq 0 \\ -2x-7 &\neq 0 \\ x &\neq -\frac{7}{2} \end{aligned}$$

we see that $x \neq -\frac{7}{2}, -3, -2$. Solving,

$$\begin{aligned} \frac{x+3}{1-\frac{3}{1-\frac{1}{x+3}}} &= -\frac{x}{2} \\ \frac{x+3}{1-\frac{3}{\left(\frac{x+3-1}{x+3}\right)}} &= -\frac{x}{2} \\ \frac{x+3}{1-\frac{3x+9}{x+2}} &= -\frac{x}{2} \\ \frac{x+3}{\left(\frac{x+2-3x-9}{x+2}\right)} &= -\frac{x}{2} \\ \frac{x+3}{\left(\frac{-2x-7}{x+2}\right)} &= -\frac{x}{2} \\ \frac{-(x+3)(x+2)}{2x+7} &= -\frac{x}{2} \\ 2(2x+7)\left(\frac{(x+3)(x+2)}{2x+7}\right) &= 2(2x+7)\left(\frac{x}{2}\right) \\ 2(x^2+5x+6) &= x(2x+7) \\ 2x^2+10x+12 &= 2x^2+7x \\ 3x+12 &= 0 \\ x &= -4 \end{aligned}$$

Therefore, $x = -4$ which satisfies $x \neq -\frac{7}{2}, -3, -2$.

b. First, we need to factor the polynomials in the denominators of each term in order to state restrictions on $x$.

$$\begin{aligned} \frac{1}{x^2-x}+\frac{1}{x^2-3x+2}+\frac{1}{x^2-5x+6}+\frac{1}{x^2-7x+12} &= \frac{1}{3} \\ \frac{1}{x(x-1)}+\frac{1}{(x-1)(x-2)}+\frac{1}{(x-2)(x-3)}+\frac{1}{(x-3)(x-4)} &= \frac{1}{3} \end{aligned}$$

So, $x \neq 0, 1, 2, 3, 4.$

If we multiply both sides of this equation by the LCD, $3x(x-1)(x-2)(x-3)(x-4)$, the resulting equation will be a 5th degree polynomial equation. Consider this alternate approach which allows for simplification as the terms on the left side of the equation are added. Add the first two fractions only.

$$\begin{aligned} \frac{1}{x(x-1)}+\frac{1}{(x-1)(x-2)} &= \frac{x-2+x}{x(x-1)(x-2)} \\ &= \frac{2(x-1)}{x(x-1)(x-2)} \\ &= \frac{2}{x(x-2)} \end{aligned}$$

Then, add the third fraction to this sum.

$$\begin{aligned} \frac{2}{x(x-2)}+\frac{1}{(x-2)(x-3)} &= \frac{2(x-3)+x}{x(x-2)(x-3)} \\ &= \frac{3(x-2)}{x(x-2)(x-3)} \\ &= \frac{3}{x(x-3)} \end{aligned}$$

Finally, add the fourth fraction to the cumulative sum.

$$\begin{aligned}\frac{3}{x(x-3)}+\frac{1}{(x-3)(x-4)} &= \frac{3(x-4)+x}{x(x-3)(x-4)}\\ &= \frac{4(x-3)}{x(x-3)(x-4)}\\ &= \frac{4}{x(x-4)}\end{aligned}$$

Thus, $\frac{4}{x(x-4)} = \frac{1}{3}$. Now multiply both sides of the equation by the LCD, $3x(x-4)$ and simplify.

$$\begin{aligned}3x(x-4)\left(\frac{4}{x(x-4)}\right) &= 3x(x-4)\left(\frac{1}{3}\right)\\ 12 &= x(x-4)\\ x^2-4x-12 &= 0\\ (x-6)(x+2) &= 0\end{aligned}$$

Therefore, $x = 6$ or $x = -2$ both of which satisfy $x \neq 0, 1, 2, 3, 4$.

13. There is no solution provided for this question.